Optimal Inapproximability of Max CSPs over large alphabet

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RANDOM-APPROX 2016

Max k-CSP$_R$

Maximum Constraint Satisfaction Problem:

- Variables take values in alphabet of size R.
- Constraints involve k variables each.
- Goal: find assignment maximizing # of satisfied constraints.

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Example

For $k = 2$, $R = 3$, a 2-CSP$_3$ is given by a list of constraints:

$$\begin{cases} (x_1 = 0 \wedge x_2 = 2) \\ (x_1 = 1 \wedge x_3 = 2) \\ \dots \end{cases}$$

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Non-boolean CSPs ($R > 2$): not resolved prior.

Hardness of Approximation

Trivial $(1/R^k)$-approximation for $\textsc{Max}$ k-CSP_R: Random assignment. Each clause matches the maximizing assignment w.p. $1/R^k$.

Q: Can we do better? Is it hard to do much better?

Prior Work: Non-boolean Max CSP

Approximation factors:[1]

	Algorithm	UG-Hardness	NP-Hardness
$k = 2$	$\frac{\log R}{R}$		
$k = 3$			
$3 \leq k < O(1)$			

[1] Ignoring constants, and for large R.

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For constant $k \geq 3$, factor of R gap in hardness vs. approximation.

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We give matching UG-hardness and approximation algorithms for any k, R. Gap reduced to $O(1)$ for constant k.[2]

[2] Original paper had polylog(R) gap. Improvement suggested by Rishi Saket, Subhash Khot, Venkat Guriswami.

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Problem

Given oracle access to $f : [R]^n \to [R]$, *determine if* f *is a dictator or "far from a dictator".*

- **Completeness** c**:** If f is a dictator, accept w.p. $\geq c$.
- **Soundness** s**:** If f is "far from" a dictator, accept w.p. $\leq s$.

"Far from dictator" $\equiv$ small low-degree influences (Fourier condition)

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$$f(x_1, x_2, \ldots x_n) := x_1 \oplus_R x_2$$

is NOT far from a dictator.

UG-Hardness of Approximation

k-query dictator test over alphabet R, with (soundness, completeness) $= (s, c)$

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$\Downarrow$

UG-hard to approximate $\textsc{Max}$ k-CSP_R better than $\approx (s/c)$.

UG-hardness of boolean 2-CSP

[Khot, Kindler, Mossel, O'Donnell]

2-Query Boolean Dictator test

$f : \{0,1\}^n \to \{0,1\}, \quad \mathbb{E}[f] = 1/2.$

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For $p \approx 0.15$,

- **Completeness:** If f is a dictator, accepts w.p. $\geq 1 - p \approx 0.85$.
- **Soundness:** If f is "far from" a dictator, accepts w.p. $\leq\approx 0.74$.

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- Ratio: $s/c \approx 0.878567 = \alpha_{GW}$

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Verifier accepts iff

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Example

majority function $maj : \{\pm 1\}^n \to \{\pm 1\}$.

$$maj(x_1, \dots, x_n) = sign(\sum_i x_i)$$

If noise η is high enough, $sign(\sum_i x_i)$ will be almost independent of $sign(\sum_i (x_i + \eta_i))$

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We show:

- **Completeness:** If f is a dictator ($f(x) = x_j$), accepts w.p. $\approx \frac{1}{(\log R)^{k/2}}$
- **Soundness:** If f is balanced and has small influences, accepts w.p. $\leq\approx \frac{1}{R^{k-1}}$

If f is far from dictator, the k queries $f(z + \eta_1), f(z + \eta_2), \ldots$ look almost independent – all equal w.p. $\approx \frac{1}{R^{k-1}}$.

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\text{(want)} &\approx \sum_{i\in[R]} \mathop{\mathbb{E}}_{x}[(g^i(x))]^k = \sum_{i} (1/R)^k = 1/R^{k-1}
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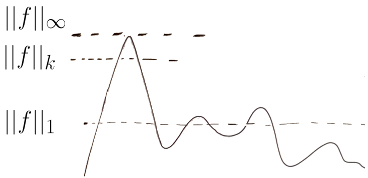

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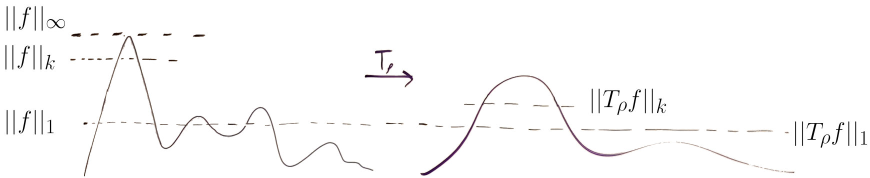

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Map a constraint

$$(X_1 = a_1) \wedge (X_2 = a_2) \wedge \cdots \wedge (X_k = a_k)$$

to all pairwise constraints $\{(X_i = a_i \wedge X_j = a_j) : 1 \leq i < j \leq k\}$

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- Tight results based on NP-hardness